

EXPANDER PROPERTY OF THE CAYLEY GRAPHS OF

$$\mathbb{Z}_m \rtimes_{\psi} \mathbb{Z}_n$$

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ABSTRACT. Let $\mathcal{T}_{m,n,k}$ denote the Cayley Graph of the split metacyclic group $\mathbb{Z}_m \rtimes_k \mathbb{Z}_n$ with respect to the symmetric generating set $S = \{(\pm 1, 0), (0, \pm 1)\}$. In this paper, we derive conditions for $\mathcal{T}_{m,n,k}$ to be a Ramanujan graph. In particular, we establish that $\mathcal{T}_{m,n,k}$ is not Ramanujan when $\min(m, n) > 8$, and hence, there are only finitely many Ramanujan graphs in the family $\{\mathcal{T}_{m,n,k}\}$. We explicitly list all possible (m, n, k) for which $\mathcal{T}_{m,n,k}$ is Ramanujan. We conclude that the graphs in the family $\{\mathcal{T}_{m,n,k}\}$ are not good spectral expanders for large m and n , which make them unsuitable for adaptation into large communication networks. Finally, we draw similar conclusions for the Cayley graphs of finite abelian groups.

1. INTRODUCTION

Interconnection networks constitute the backbone of parallel computing architecture, and the theory of Cayley graphs provide a natural framework for the design of simple, sparse, uniform, undirected, and scalable graphs. Of the multitudes of families of Cayley Graphs [11, 10], multiprocessors based on rings, toroids, and hypercubes became commercially available. Examples include Intel Hypercube, CRAY T3D with three dimensional torus, Intel Paragon with 2D mesh, and Kendall Square machines with ring topology. Optimal decentralized algorithms for various types of communication schemes like point-to-point, broadcast, multicast, and the gossip problems that exploit the topological structure of many of these networks are reported in the surveys cited above, and the references therein.

It is well known that the diameter of the network provides a lower bound on the communication algorithms, and the edge connectivity is a measure of the resilience of the network. In this paper, our goal is to analyze the expander properties of two classes of Cayley graphs based on the direct product and semi-direct product of finite cyclic groups. It is well known that rings, toroids, and hypercubes arise as Cayley graphs of direct products of suitably chosen component groups.

It is helpful to introduce some useful notations and concepts. For a finite group G and a subset S of G , let $\Gamma(G, S)$ denote the Cayley Graph of G with respect to the set S . Let $X_n = \Gamma(G_n, S_n)$ denote an infinite family of Cayley graphs indexed by the integer n . If we assume that S_n is a symmetric generating set that does not contain the identity element of G_n , then each X_n is a simple, undirected, connected, uniform graph of degree $|S_n|$. We will be interested in S_n being small and fixed, as well as one that grows slowly with n .

For any $F_n \subset G_n$, let $|E(F_n, F_n^c)|$ denote the number of edges connecting the nodes in F_n to those in its complement $S_n^c = G_n \setminus S_n$. Then the number

$$h(X_n) = \min_{F_n \subset G_n, |F_n| \leq \frac{|G_n|}{2}} \frac{|E(F_n, F_n^c)|}{|F_n|}$$

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is called the *Cheeger's constant*. $h(X_n)$ is a natural measure of the connectivity of X_n , in the sense that it measures the minimum of edges that must be removed to disconnect X_n . Clearly, $h(X_n) \leq |S_n|$, the degree of X_n . A family X_n is called as *expander* if

$$\lim_{n \rightarrow \infty} h(X_n) \geq \epsilon > 0,$$

that is $h(X_n)$ is uniformly bounded away from zero. While the problem of computing $h(X_n)$ is computationally intractable, given its importance, a number of useful lower and upper bounds on $h(X_n)$ are known. To this end, let A_n denote the adjacency matrix of X_n , and let

$$d_n = \lambda_0(X_n) > \lambda_1(X_n) \geq \dots \geq \lambda_{k_n-2}(X_n) \geq \lambda_{k_n-1}(X_n) \geq -d_n$$

be the eigenvalues of A_n , where $d_n = |S_n|$. It is well known that $\lambda_1(n) < \lambda_0(n)$ if X_n is connected, and the difference $d_n - \lambda_1(X_n)$ is called the *spectral gap*. The importance of this spectral gap stems from the fact that it is related to $h(X_n)$ as follows:

$$(*) \quad \frac{d - \lambda_1(X_n)}{2} \leq h(X_n) \leq \sqrt{2d_n(d_n - \lambda_1(X_n))}$$

Stated in other words, the family X_n is expander, if $\lim_{n \rightarrow \infty} (d_n - \lambda_1(X_n)) > 0$. This motivates the study of graphs whose λ_1 is bounded above. A d -regular graph X is called *Ramanujan* if

$$\lambda_1(X) \leq 2\sqrt{d-1}.$$

Combining this with Equation (*), we have

$$h(X_n) \geq \frac{1}{2}(d_n - 2\sqrt{d_n - 1})$$

For example, a family of graphs with constant degree $d = 2$ cannot form an expander family. Thus, Ramanujan graphs with uniform degree $d \geq 3$ are natural expanders.

In this paper, our goal is to analyze the expander property of Cayley graphs of semi-direct product $\mathbb{Z}_m \ltimes_{\psi} \mathbb{Z}_n$ of two cyclic groups. To this end, we use the well known result namely "any $d (\geq 3)$ -regular Ramanujan graphs is an expander family" [9, Chapter 3, Remark 3.12]. Using some basic concepts in the representation theory of finite groups and matrix analysis, we establish that

Main Result. *The 4-regular Cayley graphs of $\mathbb{Z}_m \ltimes_{\psi} \mathbb{Z}_n$ are not Ramanujan when $\max(m, n) > 8$, and hence are not expander graphs.*

In Section 2, we provide the basic definitions and concepts. Spectral properties of *supertoroids*, which are the Cayley graphs of semi-direct products of cyclic groups along with the main results are given in Section 3. Finally, in section 4, in addition to other concluding remarks, we show that our result on the non-expander property of toroids can be naturally extended to show that an analogous result also holds true for finite abelian groups.

2. PRELIMINARIES

In this section, we introduce some basic definitions and concepts that will be used in subsequent sections.

Definition 2.1. Let G be a finite group with identity element, and let $S \subset G$. Then the *Cayley graph of G with respect to S* , denoted by $\Gamma(G, S)$, is defined by the graph (V, E) , where

- (i) as a set, $V = G$, and
- (ii) $E = \{(g, g') : g, g' \in G \text{ and } gh = g' \text{ for some } h \in S\}$.

Remark 2.2. We state some basic properties of a Cayley graph $X = \Gamma(G, S)$.

- (i) X is a simple graph if, and only if $1 \notin S$.

- (ii) X is vertex transitive.
- (iii) X is connected if, and only if, S is a generating set.
- (iv) If S is symmetric generating set (i.e $x^{-1} \in S$, if $x \in S$), then X is a $|S|$ -regular undirected graph.

Example 2.3. For $m, n \geq 3$, consider group $G = \mathbb{Z}_m \times \mathbb{Z}_n$ and the generating set $S = \{(1, 0), (0, 1), (m-1, 0), (0, n-1)\}$. Then the 4-regular Cayley graph $\mathcal{T}_{m,n} = \Gamma(G, S)$ is called an (m, n) -toroid. The graph $\mathcal{T}_{m,n}$ can also be realised as the graph Cartesian product $C_m \square C_n$, where C_r denotes the undirected r -cycle, that is, the Cayley graph $\Gamma(\mathbb{Z}_r, \{r-1, 1\})$. Since $X \square Y$ is bipartite, if and only if both X and Y are bipartite $\mathcal{T}_{m,n}$ is bipartite only when both m and n are even. The graph $\mathcal{T}_{10,10}$ (which was generated using a basic Mathematica code [16]) is shown in Figure 1 below.

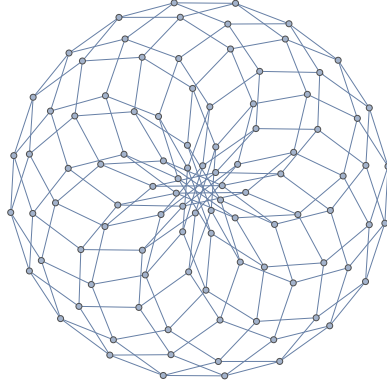


FIGURE 1. The graph $\mathcal{T}_{10,10} = C_{10} \square C_{10}$.

Notation 2.4. Let X be a k -regular graph. Then:

- (i) We shall denote the adjacency matrix of X by $A(X)$.
- (ii) The characteristic polynomial of $A(X)$ will be denoted by $p_X(\lambda)$.
- (iii) We shall denote spectrum of X by $\text{Spec}(X)$. We will use the convention that if $\lambda_0 > \lambda_1 > \dots > \lambda_{s-1}$ are the distinct eigenvalues with multiplicity $m(\lambda_i)$, for $1 \leq i \leq s-1$, then we write

$$\text{Spec}(X) = \begin{pmatrix} \lambda_0 & \lambda_1 & \dots & \lambda_{s-1} \\ m(\lambda_0) & m(\lambda_1) & \dots & m(\lambda_{s-1}) \end{pmatrix}.$$

- (iv) With the notation in (iii), we define

$$\lambda(X) = \max\{|\lambda_i| \mid 0 \leq i \leq s-1 \text{ and } |\lambda_i| \neq k\}$$

We now state some well-known properties [14] of connected k -regular graphs.

Lemma 2.5. Let X be a connected k -regular undirected graph. Then:

- (i) If λ is any eigenvalue of $A(X)$, then $|\lambda| \leq k$.
- (ii) $\lambda = k$ is always an eigenvalue of $A(X)$ with multiplicity 1.
- (iii) $\lambda = -k$ is an eigenvalue of multiplicity 1 if and only if X is bipartite.
- (iv) If $|X| = n$, then the elements of $\text{Spec}(X)$ may be arranged as

$$k = \lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{n-1} \geq -k.$$

Definition 2.6. A k -regular graph X is said to be *Ramanujan* if

$$(\dagger) \quad \lambda(X) \leq 2\sqrt{k-1}$$

The inequality (\dagger) is called the *Ramanujan condition*.

We will need the following result on the spectrum of C_n [2, Chapter 1] in order to determine whether it is Ramanujan.

Lemma 2.7. *If $X = C_n$, then the eigenvalues of $A(X)$ are given by*

$$2 \cos(2\pi k/n), 0 \leq k \leq n-1, \text{ and}$$

$$\text{Spec}(X) = \begin{cases} \begin{pmatrix} 2 & 2 \cos(2\pi/n) & \dots & 2 \cos((n-1)\pi/n) \\ 1 & 2 & \dots & 2 \end{pmatrix}, & \text{if } n \text{ is odd, and} \\ \begin{pmatrix} 2 & 2 \cos(2\pi/n) & \dots & 2 \cos((n-2)\pi/n) & -2 \\ 1 & 2 & \dots & 2 & 1 \end{pmatrix}, & \text{if } n \text{ is even.} \end{cases}$$

Example 2.8. Since $X = C_n$ is 2-regular, $2\sqrt{k-1} = 2$, and hence it follows from Lemma 2.7 that C_n is Ramanujan for any n . However, C_n is not an expander [9].

Example 2.9. The n -dimensional hypercube graph \mathcal{H}_n on $N = 2^n$ nodes is the $(\log N)$ -regular graph defined by the graph Cartesian product of n copies of C_2 . The eigenvalues of $A(\mathcal{H}_n)$ are known [7] to be $\lambda_k = n - 2k$, $0 \leq k \leq n-1$, where each λ_k has multiplicity $\binom{n}{k}$. Hence, we have that $\lambda(\mathcal{H}_n) = n-2$, which will satisfy the Ramanujan condition only when $n \leq 6$. It can be verified that \mathcal{H}_n is not an expander despite the fact that its diameter is equal to its degree ($= \log N$).

We now state a result from basic matrix analysis [1, Chapter 1], which will need later.

Lemma 2.10. *Suppose that A and B be symmetric matrices of orders m and n respectively. Let the eigenvalues of A be $\lambda_i, 1 \leq i \leq m$ and the eigenvalues of B be $\mu_j, 1 \leq j \leq n$. Then:*

(i) *The eigenvalues of the Kronecker product $A \otimes B$ are given by*

$$\lambda_i \mu_j, 1 \leq i \leq m \text{ and } 1 \leq j \leq n.$$

(ii) *The eigenvalues of the Kronecker sum $A \oplus B$ are given by*

$$\lambda_i + \mu_j, 1 \leq i \leq m \text{ and } 1 \leq j \leq n.$$

We will also need the following result from spectral graph theory [6, Chapter 9] in subsequent sections.

Lemma 2.11. *If X and Y are undirected graphs, then $A_{X \square Y} = A(X) \oplus A(Y)$.*

3. SUPERTOROIDS AND THEIR SPECTRAL PROPERTIES

In this section, we study a larger family of graphs that encompasses the family $\{\mathcal{T}_{m,n}\}$. To build these graphs, we will require an operation involving two groups that is a generalization of the direct product.

Definition 3.1. Let G and H be two groups, and let $\psi : H \rightarrow \text{Aut}(G)$ be a homomorphism. Then the *semi-direct product* $G \ltimes_{\psi} H$ of the groups G and K with respect to ϕ is the group defined by the set $G \times H$ with an operation given by

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, \psi(h_2)(h_1) h_2),$$

for any $g_1, g_2 \in G$ and $h_1, h_2 \in H$.

Remark 3.2. When ψ is trivial, $G \ltimes_{\psi} H$ yields direct product $G \times H$, and hence the semi-direct product is a generalization of the direct product. Moreover, $G \ltimes_{\psi} H$ is abelian if, and only if, both G and H are abelian and ψ is trivial.

Example 3.3. Suppose that $G = \mathbb{Z}_m$ and $H = \mathbb{Z}_n$. Then it is well-known that $\text{Aut}(\mathbb{Z}_n) \cong U_n$, the multiplicative group of integers modulo n , whose order is given by the Euler totient function $\phi(n)$. So a nontrivial homomorphism $\psi : \mathbb{Z}_m \rightarrow U_n$ will exist if, and only if $\gcd(m, \phi(n)) > 1$. Such a homomorphism is given by $1 \xrightarrow{\psi} (1 \mapsto k)$, where $k^m \equiv 1 \pmod{n}$ [13, Chapter XII, §2]. Hence, the group operation in $\mathbb{Z}_m \ltimes_{\psi} \mathbb{Z}_n$ takes the form

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 + g_2, \psi(1)^{g_2}(h_1) + h_2) = (g_1 + g_2, k^{g_2} h_1 + h_2),$$

for any $g_1, g_2 \in G$ and $h_1, h_2 \in H$. As ψ depends entirely on k , this semi-direct product is often written as $\mathbb{Z}_m \ltimes_k \mathbb{Z}_n$, which is the notation we will follow.

Remark 3.4. The *dihedral groups* given by $D_{2n} = \mathbb{Z}_2 \ltimes_{n-1} \mathbb{Z}_n$, which can be realized as the group of symmetries of a regular n -gon, form an important subfamily of $\{\mathbb{Z}_m \ltimes_k \mathbb{Z}_n\}$. Moreover, the family $\{\mathbb{Z}_m \ltimes_k \mathbb{Z}_n\}$ is a subfamily of a larger family of groups known as *metacyclic groups*, which were completely classified by Hempel [8].

Definition 3.5. The Cayley graph $\mathcal{T}_{m,n,k} = \Gamma(\mathbb{Z}_m \ltimes_k \mathbb{Z}_n, S)$, where

$$S = \{(1, 0), (0, 1), (m-1, 0), (0, n-1)\}$$

is called an (m, n) -*supertoroid*.

Note that $\mathcal{T}_{m,n,k}$ is a connected 4-regular graph and when $k = 1$, it is equal to the graph $\mathcal{T}_{m,n}$.

Example 3.6. When $m = 4$ and $n = 8$, there are four possible solutions to the equation $k^m \equiv 1 \pmod{n}$, namely $k = 1, 3, 5$, and 7 . However, it can be shown that $\mathbb{Z}_4 \ltimes_1 \mathbb{Z}_8 \cong \mathbb{Z}_4 \ltimes_7 \mathbb{Z}_8$ and $\mathbb{Z}_4 \ltimes_3 \mathbb{Z}_8 \cong \mathbb{Z}_4 \ltimes_5 \mathbb{Z}_8$. Consequently, up to isomorphism, there are only two distinct graphs in the family $\{\mathcal{T}_{4,8,k}\}$, which are the toroid $\mathcal{T}_{4,8,1} = \mathcal{T}_{4,8}$ and the supertoroid $\mathcal{T}_{4,8,3}$. The graph $\mathcal{T}_{4,8,3}$ is illustrated in Figure 2 below.

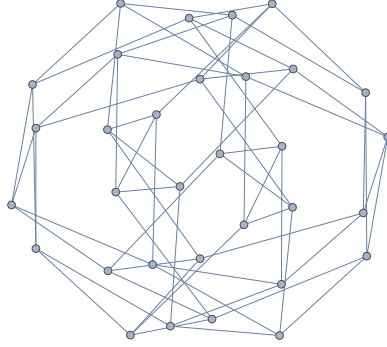


FIGURE 2. The supertoroid $\mathcal{T}_{4,8,3}$.

3.1. Spectra of toroids. We begin by understanding the spectra of the subfamily $\mathcal{T}_{m,n}$. Since $\mathcal{T}_{m,n} = C_m \square C_n$, the following theorem follows immediately from Lemma 2.11.

Theorem 3.7. Let $X = \mathcal{T}_{m,n}$. Then

$$A(X) = A(C_m) \oplus A(C_n).$$

We can now compute $\text{Spec}(\mathcal{T}_{m,n})$ directly using Theorem 3.7 and Lemmas 2.7, 2.10 and 2.11.

Corollary 3.8. *If $X = \mathcal{T}_{m,n}$, then eigenvalues of $A(\mathcal{T}_{m,n})$ are given by*

$$2(\cos(2\pi k/m) + \cos(2\pi \ell/n)), 0 \leq k \leq \ell - 1, 0 \leq \ell \leq n - 1.$$

Moreover, each eigenvalue λ has multiplicity given by

$$m(\lambda) = \begin{cases} 1, & \text{if } |\lambda| = 4, \text{ and} \\ 2, & \text{otherwise.} \end{cases}$$

Another consequence of Theorem 3.7 is the following property of the family $\{\mathcal{T}_{m,n}\}$.

Corollary 3.9. *The graph $\mathcal{T}_{m,n}$ is not Ramanujan when $\max(m, n) > 8$.*

Proof. Let $X = \mathcal{T}_{m,n}$, then from Theorem 3.8, it is apparent that $\lambda(X) = \max\{2 + 2\cos(2\pi/m), 2 + 2\cos(2\pi/n)\}$. Assuming without loss of generality that $m \geq n$, we have $\lambda(X) = 2 + 2\cos(2\pi/m)$. This implies that for $m > 8$,

$$\lambda(X) > 2 + \sqrt{2} > \sqrt{3},$$

which clearly violates the Ramanujan condition. \square

Remark 3.10. It is interesting to note that $\mathcal{T}_{8,8}$ is in fact a Ramanujan graph. Using a Mathematica code [16], we were able to compute that there are precisely 18 Ramanujan graphs in the family $\{\mathcal{T}_{m,n}\}$, which we will list in the following section.

3.2. Spectra of supertoroids. To understand the spectra of general supertoroids, we will need to make a short detour through the theory of representations of finite groups [5, 15].

Definition 3.11. Let G be a finite group, and let $\text{GL}(W)$ denote the group of all invertible linear maps on a finite-dimensional complex vector space W . Then a homomorphism $\pi : G \rightarrow \text{GL}(W)$ is called a *representation* of G .

The dimension of the vector space W is called the *dimension of the representation*. A representation as in Definition 3.11 is often abbreviated as (π, W) .

Definition 3.12. A representation (π, W) of a group G is said to be *faithful* if π is an injective map.

Example 3.13. For a finite group G , let W be the vector space of all complex valued functions on G . Then the homomorphism $\pi : G \rightarrow \text{GL}(W)$ defined by

$$\pi(g)(f)(h) = f(g^{-1}(h))$$

is a faithful $|G|$ -dimensional representation of the group G called the *regular representation* of G .

Definition 3.14. Let (π, W) be a representation of a group G . Consider the homomorphism $\chi_{\pi} : G \rightarrow \text{GL}(\mathbb{C})(\cong \mathbb{C}^{\times} = \mathbb{C} \setminus \{0\})$ defined by

$$\chi_{\pi}(g) = \text{tr}(\pi(g)),$$

for all $g \in G$, where $\text{tr}(A)$ denotes the trace of a square matrix A . Then the representation (χ_{π}, \mathbb{C}) is called the *character* of the representation (π, W) .

In fact, any one-dimensional representation of a group G is also called a *character* of G .

Definition 3.15. Let (π, W) be a representation of a group G . Then a subspace U of W is said to be *G-invariant* if $\pi(g)U = U$, for all $g \in G$.

Definition 3.16. A representation (π, W) of a group G is *irreducible* if there exists no nontrivial G -invariant subspace U of W .

Note that by definition every character is irreducible. We now state a result from basic representation theory.

Lemma 3.17. *Let (π, W) be a representation for a finite group G . Then:*

- (i) (π, W) is a direct sum of irreducible representations.
- (ii) If G is abelian and (π, W) is irreducible, then (π, W) is one-dimensional (or a character).

Example 3.18. By Lemma 3.17, every irreducible representation (χ, W) of \mathbb{Z}_n is one-dimensional. Suppose that $\Omega_n = \{1, \omega, \omega^2, \dots, \omega^{n-1}\}$ denotes multiplicative group of the complex n^{th} roots of unity. Then there are exactly n homomorphisms $\mathbb{Z}_n \rightarrow \mathbb{C}^\times$, namely the homomorphisms given by $\chi_k(i) = \omega^{ki}$, for $0 \leq k \leq n-1$. The representations (χ_k, \mathbb{C}) , for $0 \leq k \leq n-1$, are precisely the characters of \mathbb{Z}_n .

In the following example, we describe the regular representation of \mathbb{Z}_n .

Example 3.19. Consider any faithful representation (π, V) of \mathbb{Z}_n of dimension n . Then by definition, $\pi : \mathbb{Z}_n \rightarrow \text{GL}(V) (\cong \text{GL}_n(\mathbb{C}))$ is an injective homomorphism. Since π is completely determined by $\pi(1)$, if $\pi(1) = A \in \text{GL}_n(\mathbb{C})$, then $A^n = 1$. Conversely, given any $A \in \text{GL}(n, \mathbb{C})$ such that $A^n = 1$, the map $\varphi_A : \mathbb{Z}_n \rightarrow \text{GL}_n(\mathbb{C})$ defined by $\varphi_A(1) = A$ yields an n -dimensional representation for \mathbb{Z}_n . Thus every n -dimensional of \mathbb{Z}_n is of this form. To compute the regular representation ψ of \mathbb{Z}_n , we first identify \mathbb{Z}_n with Ω_n as in Example 3.18. As the regular representation of a group is analogous to its self-action, it follows that

$$\psi(k) = (a_{ij})_{n \times n}, \text{ where } a_{ij} = \delta_{\omega^j, \omega^i \omega^k} \text{ and } \omega = e^{i2\pi/n}.$$

In other words,

$$a_{ij} = \begin{cases} 1, & \text{if } j - i \equiv k \pmod{n}, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3.20. Let G be a finite group, and let $A_{(G,g)} = A(\Gamma(G, \{g\}))$. Then $\psi : G \rightarrow \text{GL}_{|G|}(\mathbb{Q})$ defined by $\psi(g) = A_{(G,g)}$, for all $g \in G$, is a faithful representation [4] called the *adjacency representation* of G .

From the Definition 3.20 above, it is clear that adjacency representation of a group G is indeed isomorphic to its regular representation.

Remark 3.21. When $G = \mathbb{Z}_n$, it follows from Example 3.19 that $A_{(G,k)} = \psi(k)$, for any $k \in \mathbb{Z}_n$.

The following lemma [4, Theorem 3.2] will be used in subsequent results.

Lemma 3.22. *Let G be a finite group and $S \subset G$. Then*

$$A(\Gamma(G, S)) = \sum_{g \in S} A_{(G,g)}.$$

Lemma 3.23. *If $X = C_n$, then*

$$A(X) = \begin{cases} 1, & \text{if } |j - i| \equiv 1 \pmod{n}, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let C_n^+ denote the counter-clockwise oriented n -cycle and C_n^- denoted the clockwise oriented n -cycle. Then $A(C_n^+) = A_{(\mathbb{Z}_n, 1)}$ and $A(C_n^-) = A_{(\mathbb{Z}_n, n-1)} = A(C_n^+)^T$. From Lemma 3.22, we can infer that $A(C_n) = A(C_n^+) + A(C_n^-)$. The result now follows from Remark 3.21. \square

Fixing the notation in Lemma 3.23, we now prove the main result in this paper, which gives an explicit description $A(\mathcal{T}_{m,n,k})$.

Theorem 3.24. *Let $X = \mathcal{T}_{m,n,k}$. Then $A(X) = U + V$, where $U = A(C_m) \otimes I_n$ and $V = (v_{ij})$ is an $m \times m$ block matrix whose $n \times n$ blocks are given by*

$$v_{ij} = \begin{cases} A(C_n^+)^{k^{i-1}} + A(C_n^-)^{k^{i-1}}, & \text{if } i = j, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let X^+ be the Cayley Graph $\Gamma(\mathbb{Z}_m \ltimes_k \mathbb{Z}_n, \{(1, 0), (0, 1)\})$ and X^- be the Cayley Graph $\Gamma(\mathbb{Z}_m \ltimes_k \mathbb{Z}_n, \{(m-1, 0), (0, n-1)\})$. Then by Lemma 3.22, we can infer that $A(X) = A(X^+) + A(X^-)$. Now by applying [4, Theorem 3.3], we have $A(X^+) = U^+ + V^+$, where $U^+ = (u_{ij}^+)$ is an $m \times m$ block matrix whose $n \times n$ blocks are given by

$$u_{ij}^+ = \begin{cases} I_n, & \text{if } j - i \equiv 1 \pmod{m}, \text{ and} \\ 0, & \text{otherwise,} \end{cases}$$

and $V^+ = (v_{ij}^+)$ is an $m \times m$ block matrix whose $n \times n$ blocks are given by

$$v_{ij}^+ = \begin{cases} A(C_n^+)^{k^{i-1}}, & \text{if } i = j, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, $A(X^-) = U^- + V^-$, where $U^- = (u_{ij}^-)$ is an $m \times m$ block matrix whose $n \times n$ blocks are given by

$$u_{ij}^- = \begin{cases} I_n, & \text{if } j - i \equiv -1 \pmod{m}, \text{ and} \\ 0, & \text{otherwise,} \end{cases}$$

and $V^- = (v_{ij}^-)$ is an $m \times m$ block matrix whose $n \times n$ blocks are given by

$$v_{ij}^- = \begin{cases} A(C_n^-)^{k^{i-1}}, & \text{if } i = j, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Finally, we obtain the required result by observing that $U^+ = A(C_m^+) \otimes I_n$ and $U^- = A(C_m^-) \otimes I_n$. □

Note that taking $X = \mathcal{T}_{m,n,1} = \mathcal{T}_{m,n}$ in Theorem 3.24 simply yields Theorem 3.7.

In order to predict the Ramanujan behavior of $X = \mathcal{T}_{m,n,k}$, we will need an upper bound on $\lambda(X)$. We shall appeal to the following result from matrix analysis [1, Theorem III.2.1] to derive such a bound. For the remaining part of the paper, we shall use $\lambda_i(M)$, $1 \leq i \leq n$, to denote the eigenvalues of an $n \times n$ symmetric matrix M arranged in descending order.

Lemma 3.25. *If U and V are $n \times n$ symmetric matrices, then for $i \leq j$,*

$$\lambda_j(U + V) \leq \lambda_i(U) + \lambda_{j-i+1}(V).$$

Corollary 3.26 (Main result). *Let $X = \mathcal{T}_{m,n,k}$ and $M = \max(m, n)$. Then*

$$\lambda(X) \leq 2 + 2 \cos(2\pi/M), \text{ and}$$

consequently, X is not Ramanujan when $M > 8$.

Proof. First, we note that the eigenvalues of $A(C_n^+)$ are $e^{i2\pi j/n}$, while those of $A(C_n^-)$ are $e^{-i2\pi j/n}$, where $0 \leq j \leq n-1$. Since the matrices $A(C_n^+)$ and $A(C_n^-) = A(C_n^+)^{\top}$ satisfy

$$A(C_n^+)A(C_n^-) = A(C_n^-)A(C_n^+) = 0,$$

by a version of the Spectral Theorem [12, Chapter VIII], they are simultaneously diagonalizable. Hence, their powers are also simultaneously diagonalizable, and so the eigenvalues of the matrix V in Theorem 3.24 are given by

$$e^{i2\pi k^i j/n} + e^{-i2\pi k^i j/n} = 2 \cos(2\pi k^i j/n), \quad 0 \leq i \leq m-1 \text{ and } 0 \leq j \leq n-1.$$

From this, we can infer that $\lambda_1(V) = 2$. On the other hand, by Lemma 2.10, U has m distinct eigenvalues, each of degree n , which are given by

$$2 \cos(2\pi k/m), \quad 0 \leq k \leq m-1.$$

Finally, we apply Lemma 3.25 to get

$$\lambda(X) = \lambda_2(X) \leq \lambda_2(U) + \lambda_1(V) = 2 + 2 \cos(2\pi/m),$$

and interchanging the roles of U and V yields the same inequality, but with the m replaced by n . The result now follows from these two inequalities. \square

Using our Mathematica software, we were able to compute that there are precisely 31 Ramanujan graphs in the family $\{\mathcal{T}_{m,n,k}\}$. Below, we have listed all triples (m, n, k) for which $\mathcal{T}_{m,n,k}$ is Ramanujan.

$$\begin{aligned} &(3, 2, 1), (3, 3, 1), (4, 2, 1), (4, 3, 1), (4, 3, 2), (4, 4, 1), (4, 4, 3), \\ &(5, 3, 1), (5, 5, 1), (6, 2, 1), (6, 3, 1), (6, 3, 2), (6, 4, 1), (6, 4, 3), \\ &(6, 6, 1), (6, 6, 5), (7, 3, 1), (7, 5, 1), (8, 2, 1), (8, 3, 1), (8, 3, 2), \\ &(8, 4, 1), (8, 4, 3), (8, 5, 2), (8, 5, 3), (8, 6, 1), (8, 6, 5), (8, 8, 1), \\ &\quad (8, 8, 3), (8, 8, 5), (8, 8, 7) \end{aligned}$$

Note that this list also encompasses the 18 Ramanujan toroids that we had mentioned earlier.

4. CONCLUDING REMARKS

4.1. Conclusion. From the discussion in the preceding section, we now know that there are only finitely many Ramanujan graphs in the infinite family $\{\mathcal{T}_{m,n,k}\}$, in which every graph with $k \neq 1$ represents a non-abelian group. Keeping in mind the fact that abelian groups yield poor expander graphs [9, Chapter 8], the existence of only a few desirable graphs in such a large family of non-abelian graphs is certainly striking. Clearly, this is not an expander family of graphs, and hence we can safely conclude that it is not suited for adaptation into large networks.

4.2. Finite abelian groups. We can further expand this discussion, by generalizing 2-dimensional family of graphs $\{\mathcal{T}_{m,n}\}$ to higher dimensions in the following manner. For a set $K = \{m_1, \dots, m_n\}$ of positive integers, let

$$\mathcal{T}_S = C_{m_1} \square \dots \square C_{m_n}.$$

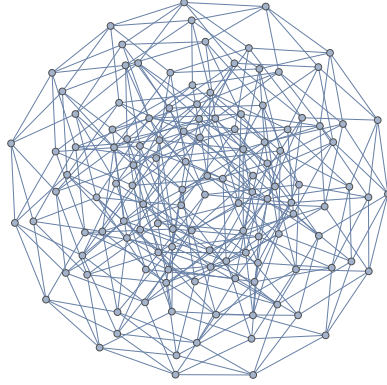
\mathcal{T}_S is the $2n$ -regular Cayley graph $\Gamma(G_K, S)$, where

$$G_K = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \dots \times \mathbb{Z}_{m_n} \text{ and } S = \prod_{i=1}^n \{m_i - 1, 1\}.$$

as shown in Figure 3 below. By an inductive application of Lemma 3.25, we see that

$$\lambda(\mathcal{T}_S) = 2(n-1) + 2 \cos(2\pi/M), \text{ where } M = \max\{m_1, \dots, m_n\}.$$

From this, we can infer that when M and n are sufficiently large, X will be not be a Ramanujan graph. The classification of finite abelian groups [3, Chapter 5] states that every finite abelian group is isomorphic to a group of type G_K . Therefore, by a direct extension of our methods, we have established that finite abelian groups also yield non-expander families of Cayley graphs.

FIGURE 3. The graph $C_5 \square C_5 \square C_5$.

4.3. Conjectures. Based on the observations made in this paper, we believe the following natural generalization of our result should hold true.

Conjecture 4.1. *For a given positive integer $n \geq 3$, the Cayley graphs of all possible semi-direct products involving n cyclic groups will form a non-expander family.*

An obvious hurdle with such a generalization is the non-associativity of these products. Moreover, there are several other families of metacyclic groups whose Cayley graphs could be analyzed for their expander properties. However, we believe that in general:

Conjecture 4.2. *Metacyclic groups will yield poor expander families of Cayley graphs.*

It would also be an interesting endeavor to investigate the expander properties of finite “nearly abelian” groups, otherwise known as *nilpotent groups*.

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